Simple groups and the number of countable models

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Abstract

Let T be a complete, superstable theory with fewer than $2^{\aleph_0}$ countable models. Assuming that generic types of infinite, simple groups definable in T^{eq} are sufficiently non-isolated we prove that ω^ω is the strict upper bound for the Lascar rank of T.

Throughout the paper T is a complete, superstable theory in a countable language having infinite models. $I(T,\aleph_0)$ is the number of its countable models. $S_n(T)$ is the space of all complete n-types and $S(T)=\bigcup_{n\in\omega}S_n(T)$. U denotes Lascar's rank of complete types and U-rank of the theory is $U(T)=\sup\{U(p)\,|\,p\in S(T)\}$. In [8] it was conjectured:

Conjecture 1. $U(T)\geq\omega^\omega$ *implies* $I(T,\aleph_0)=2^{\aleph_0}$.

There the conjecture was proved for trivial theories and for one-based theories, but the general case is still open even for $\aleph_0$-stable theories. The proof in [8] was based on a technical fact (see Proposition 1.1 below) asserting that whenever in a superstable theory there exists an infinite family of sufficiently non-isolated types $\{p_n\,|\,n\in I\}$ such that each p_n has a finite domain and U-rank equal to ω^n, then $I(T,\aleph_0)=2^{\aleph_0}$ holds; here 'sufficiently non-isolated' refers to 'eventually strongly non-isolated', or ESN for short, which is defined below. So, in order to prove the conjecture, assuming $U(T)\geq\omega^\omega$ it suffices to find an infinite family of ESN types p_n with $U(p_n)=\omega^n$. This was easily done in [8] because in a one-based or trivial theory any type of limit-ordinal U-rank turned out to be ESN. In this article we will show that the nonexistence of such a family in the general case is of geometric nature:

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it is caused by a presence of simple groups of U-rank $\omega^n \cdot k$ in T^{eq} where n can be arbitrarily large natural number. Also, we will prove that the generic type of a field of U-rank $\omega^n \cdot k$ is ESN, so these simple groups are 'big-bad': they do not interpret a field of approximately the same U-rank as that of the group.

The situation is clear in the finite rank case: it is well known that any simple group of finite rank is $\aleph_1$-categorical but not $\aleph_0$-categorical. This implies that its generic type is eventually non-isolated, meaning that its nonforking extension over some finite set is non-isolated; it is also ESN, because the two notions coincide for types of rank 1. It is interesting whether the generic type of an arbitrary superstable group is eventually nonisolated.

Question 1. *Is the generic type of any simple, superstable group eventually non-isolated?*

The main result of this article is:

Theorem 1. *If T is superstable, $U(T) \geq \omega^\omega$ and the generic type of any simple group definable in T^{eq} of U-rank smaller than ω^ω is eventually strongly non-isolated then $I(T, \aleph_0) = 2^{\aleph_0}$.*

Theorem 1 is a simplified and corrected version of the corresponding result from the author's PhD Thesis [9].

1 Preliminaries

We will assume that the reader is familiar with basic stability theory and stable group theory, references are [1], [4], [5], [6] and [10]. Throughout the paper we will assume $T = T^{eq}$ and operate in the monster model $\bar{M}$ of T. The notation is standard. A regular type is assumed to be stationary. For any regular type $p \in S(A)$ and any B by $\mathrm{cl}_p(B)$ we will denote the set of elements realizing a forking extension of p over AB. This is a pregeometry operator on the locus of p. If p, q are possibly incomplete types then p is *q-internal* if whenever M is $\aleph_1$-saturated and contains $\mathrm{dom}(p) \cup \mathrm{dom}(q)$ then for any a realizing p there is a tuple $\bar{b}$ of realizations of q such that $a \in \mathrm{dcl}(\bar{b}M)$. The binding group is the group of all automorphisms of $p(\bar{M})$ fixing pointwise $\mathrm{dom}(p) \cup \mathrm{dom}(q) \cup q(\bar{M})$; if p, q are stationary and p is q-internal then the binding group is type-definable.

$p \in S(A)$ is *semiregular* if it is stationary and there is a regular q such that p is q-simple and domination-equivalent to a power of q, in which case we also say that p is *q-semiregular*. If $\mathrm{tp}(\bar{a}/A) \not\perp q$ then there is $b \in \mathrm{acl}(\bar{a}A) \setminus \mathrm{acl}(A)$

such that $\mathrm{stp}(b/A)$ is q-internal; if in addition $\mathrm{tp}(\bar{a}/A)$ is stationary, then such a b can be found in $\mathrm{dcl}(\bar{a}A) \setminus \mathrm{dcl}(A)$. Moreover, if $\omega^\alpha \cdot n$ is the lowest monomial term in the Cantor normal form of $U(\bar{a}/A)$ then there is a regular $q \not\perp \mathrm{tp}(\bar{a}/A)$ of U-rank ω^α and for the corresponding b we have $U(b/A) = \omega^\alpha \cdot m$ where $m \leq n$; in particular $\mathrm{stp}(b/A)$ is q-semiregular and q-internal. In this article we will often deal with types which are q-semiregular and q-internal for some regular q of U-rank ω^α; their U-rank is the monomial $\omega^\alpha \cdot m$ where $m = \mathrm{wt}_q(b/A)$, and any extension of such a type of U-rank at least ω^α is $\not\perp q$.

Recall that $p \in S(A)$ is *eni*, or eventually non-isolated, iff there is a finite set B and a non-isolated, nonforking extension of p in $S(AB)$. p is ENI if it is strongly regular and eni; p is NENI if it is strongly regular and is not eni (this slightly differs from the original definition from [7] in that we allow a NENI type to have infinite domain).

Next we recall the notion of strong non-isolation from [8]. Let $p \in S(A)$ be non-algebraic. p is *strongly nonisolated* if for all n and all finite B

$$\{q \in S_n(AB) \mid q \overset{a}{\perp} p\} \quad \text{is dense in } \; S_n(AB);$$

here $p \overset{a}{\perp} q$ denotes almost orthogonality: any pair of realizations of $p \mid AB$ and q is independent over AB. Note that a strongly non-isolated type is almost orthogonal to all isolated types; in particular, it is non-isolated. Moreover, if T is small (i.e. $|S(\emptyset)| = \aleph_0$) then isolated types are dense in $S_n(A)$ for any finite A and strong non-isolation of $p \in S(A)$ is equivalent to: p is almost orthogonal to any isolated type over a slightly larger domain. p is *eventually strongly nonisolated*, or ESN for short, if there is a finite B and a nonforking extension $q \in S(AB)$ which is strongly nonisolated. By Theorem 1 from [8] we have:

Theorem 2. *(T countable superstable) $p \in S(A)$ is ESN if and only if it is orthogonal to all NENI types whose domain is a finite extension of A in $\bar{M}^{eq}$.*

This is a strong dichotomy especially for regular types over finite domains: such a type is either ESN or is $\not\perp$ to a NENI type; if T is $\aleph_0$-stable, then it coincides with the ENI-NENI dichotomy. A consequence of the theorem is that the property of being ESN is preserved under non-orthogonality, for regular types whose domains differ on a finite set. Also, by Proposition 2.1 from [8], a type $p \in S(A)$ is ESN if and only if each of its regular components is ESN, assuming that the domain of each component differs from A on a finite set.

The next fact is an instance of Proposition 5.1 from [8]:

Proposition 1.1. *Suppose that there exists an infinite $I \subseteq \omega$ and a family $\{p_n | n \in I\}$ of regular, ESN types over finite domains such that $U(p_n) = \omega^n$ for all $n \in I$. Then $I(T, \aleph_0) = 2^{\aleph_0}$.*

2 Internally isolated types

The notion of internal isolation for types was introduced in [8] in order to approximate certain definability property of forking on the locus of a NENI type: If A is finite and $p \in S(A)$ is NENI then any nonforking extension of p over a finitely extended domain is isolated; by induction, it is not hard to prove that p^n is isolated for all n.

Definition 2.1. *A stationary type $p \in S(A)$ is internally isolated if for each $n \in \mathbb{N}$ there exists a formula $\phi_n(x_1, x_2, ..., x_n)$ over A such that:*

$$(p(x_1) \wedge p(x_2) \wedge ... \wedge p(x_n) \wedge \phi_n(x_1, x_2, ..., x_n)) \Leftrightarrow p^n(x_1, x_2, ..., x_n).$$

Another way to describe internal isolation of $p \in S(A)$ is the following:

for all n the locus of $p^n(\bar{M})$ is a relatively A-definable subset of $p(\bar{M})^n$;

where a subset of a type-definable over A set C is *relatively A-definable* if it is the intersection of C and an A-definable set. Here we also note that, by Lemma 1.2 from [8], a complete type is NENI if and only if it is regular, isolated and internally isolated.

In the next lemma we prove that internal isolation of a regular type p has a strong consequence: relative definability of cl_p within $p(\bar{M})^n$.

Lemma 2.2. *Suppose that $p \in S(A)$ is regular and internally isolated. Then*

$$\{(a, b_1, ..., b_n) \in p(\bar{M})^{n+1} \,|\, a \in \mathrm{cl}_p(\bar{b})\}$$

is a relatively A-definable subset of $p(\bar{M})^{n+1}$ for all n

Proof. In order to simplify notation we will assume that $p \in S_1(A)$. Fix n and let $S \subset S_{n+1}(A)$ be the set of all completions of $p(x) \cup p(y_1) \cup ... \cup p(y_n)$. We will prove that $C = \{\mathrm{tp}(a\bar{b}/A) \in S \,|\, a \in \mathrm{cl}_p(\bar{b})\}$ is clopen in S.

Suppose that $\mathrm{tp}(a\bar{b}/A) \in C$. Then there is an independent over A set $B \subset \bar{b}$ such that $a \in \mathrm{cl}_p(B)$; without loss of generality we will assume that $B = b_1...b_m$. Note that $b_1...b_m \models p^m$ and that $ab_1...b_m$ does not realize p^{m+1}. Consider the formula:

$$\neg\phi_{m+1}(x, y_1, ..., y_m) \wedge \phi_m(y_1, ..., y_m)$$

(where ϕ_i's are given by Definition 2.1). It belongs to $\mathrm{tp}(a\bar{b}/A)$ and whenever $\mathrm{tp}(a'\bar{b}'/A) \in S$ contains the formula then $a' \in \mathrm{cl}_p(\bar{b}')$. Therefore C is open is S.

To prove that $S \smallsetminus C$ is open in S suppose that $\mathrm{tp}(c\bar{b}/A) \in S \smallsetminus C$. Then $c \models p \,|\, A\bar{b}$. Choose a maximal independent subset of $\bar{b}$ over A; without loss of generality suppose that $\{b_1, ..., b_k\}$ is chosen. Consider the formula

$$\phi_{k+1}(x, y_1, ..., y_k) \wedge \bigwedge_{i=k+1}^{n} \neg\phi_{k+1}(y_i, y_1, ..., y_k)\,.$$

Clearly it belongs to $\mathrm{tp}(a\bar{b}/A)$, and whenever $\mathrm{tp}(a'\bar{b}'/A) \in S$ contains the formula then $a'b_1'...b_k' \models p^{k+1}$ and $b_i' \in \mathrm{cl}_p(b_1'...b_k')$ holds for all $k < i \leq n$. Combining the two we derive $a' \notin \mathrm{cl}_p(\bar{b}')$, so our formula witnesses that $S \smallsetminus C$ is open in S. This completes the proof of the lemma. □

As an immediate corollary we obtain:

Corollary 2.3. *Suppose that $p \in S(A)$ is regular and internally isolated. Then $\mathrm{cl}_p(\bar{y}) = \mathrm{cl}_p(\bar{z})$ is a relatively A-definable equivalence relation on $p(\bar{M})^n$ for all n.*

By compactness, it follows that for any regular, internally isolated type $p \in S(A)$ there exists a formula over A defining an equivalence relation on the whole of $\bar{M}^n$ and relatively defining $\mathrm{cl}_p(\bar{y}) = \mathrm{cl}_p(\bar{z})$ within $p(\bar{M})^n \times p(\bar{M})^n$.

Definition 2.4. *Suppose that $p \in S(A)$ is regular and internally isolated.*

(1) $E_n^p(\bar{y}, \bar{z})$ is a formula defining an equivalence relation on the whole of M^n and relatively defining $\mathrm{cl}_p(\bar{y}) = \mathrm{cl}_p(\bar{z})$ on $p(\bar{M})^n$.

(2) $p_{(n)} = p^n / E_n^p$.

Throughout the paper whenever the meaning of p is clear from the context then we will simply write E_n instead of E_n^p. Further, note that $p_{(n)}$ is a complete type over A, we will refer to it as to the type of the name of an n-dimensional p-subspace (Grassmannian).

Remark 2.5. *Suppose that $p \in S(A)$ is regular and internally isolated. Let $\bar{a} \models p^n$ and let $c = \bar{a}/E_n$.*

(i) There is a unique type over cA of an n-tuple of members of $\mathrm{cl}_p(\bar{a})$ which are independent over A. In other words, if $\bar{b} \models p^n$ and $\bar{b} \subseteq \mathrm{cl}_p(\bar{a})$ then $\mathrm{tp}(\bar{a}/cA) = \mathrm{tp}(\bar{b}/cA)$. This holds because any A-automorphism moving $\bar{a}$ to $\bar{b}$ fixes setwise $\mathrm{cl}_p(\bar{a})$ $(= \mathrm{cl}_p(\bar{b}))$, so it is an Ac-automorphism.

(ii) For $m \leq n$ any independent over A m-tuple is contained in an independent n-tuple so, by part (i), there is a unique type over cA of an independent (over A) m-tuple of elements of $\mathrm{cl}_p(\bar{a})$.

(iii) p has a unique forking extension in $S(cA)$: applying part (ii) to the case $m = 1$ we conclude that there is a unique extension of p in $S(cA)$ consistent with $x \in \mathrm{cl}_p(\bar{a})$; it is a forking extension because the nonforking extension clearly satisfies $x \perp \bar{a}\,(A)$.

(iv) The uniqueness of forking extension implies that $\mathrm{cl}_p(\bar{a}) = \mathrm{cl}_p(c)$ holds.

Definition 2.6. *Suppose that $p \in S(A)$ is regular and internally isolated, and that $c \models p_{(n)}$. By p_c we will denote the unique forking extension of p in $S_1(cA)$.*

Thus p_c is the type of an element of the subspace c. Next we recall Definition 2 from [8]: a regular type $p \in S(A)$ is *strictly regular* if whenever $a_1, a_2 \models p$ then either $a_1 = a_2$ or $a_1 \perp a_2\,(A)$ holds.

Lemma 2.7. *Suppose that $p \in S(A)$ is regular and internally isolated.*

(i) $p_{(1)}$ is strictly regular :

(ii) $U(p_{(1)}) = \omega^\alpha$ where α is the smallest power of a monomial in the Cantor normal form of $U(p)$.

Proof. (i) Suppose that a_1, a_2 realize $p_{(1)}$. Choose $b_1, b_2 \models p$ such that $b_i/E_1 = a_i$. Then $a_1 \not\perp a_2\,(A)$ implies $b_1 \not\perp b_2\,(A)$ and $\models E_1(b_1, b_2)$ holds. Thus $b_1/E_1 = b_2/E_2$ and $a_1 = a_2$.

(ii) Let a_1 realize $p_{(1)}$. Since p is regular there is $b \in \mathrm{dcl}(a_1A) \setminus \mathrm{acl}(A)$ such that $U(b/A) = \omega^\alpha$. Let a_1, a_2 be a Morley sequence in $\mathrm{stp}(a_1/bA)$. Then $a_1 \not\perp b\,(A)$ and $a_2 \not\perp b\,(A)$ imply, by regularity, $a_1 \not\perp a_2\,(A)$. By part (i) we get $a_1 = a_2$. Since a_1, a_2 is a Morley sequence we conclude $a_1 \in \mathrm{dcl}(bA)$ and $U(a_1/A) = \omega^\alpha$. □

We order $\cup_{n \geq 1} p_{(n)}(\bar{M})$ by inclusion of cl_p-closures: $c \leq c'$ iff $\mathrm{cl}_p(c) \subseteq \mathrm{cl}_p(c')$.

Lemma 2.8. *Suppose that $p \in S_1(A)$ is regular and internally isolated. If $c \leq c'$ realize $p_{(n)}$'s and $a \models p_c$ then $a \perp c'(cA)$.*

Proof. $c \leq c'$ implies that there are $a_1, ..., a_m, ..., a_n \models p^n$ such that:

$$a_1...a_m/E_m = c \text{ and } a_1...a_n/E_n = c'.$$

Clearly, any automorphism of $\bar{M}$ fixing $c, a_{m+1}...a_n$ pointwise fixes also c', so $c' \in \mathrm{dcl}(ca_{m+1}...a_nA)$. From the independence of $a_1, ..., a_n$ and $c \in \mathrm{dcl}(a_1, ..., a_m, A)$ we get $a_1 \perp a_{m+1}...a_n\,(cA)$. Combining with $c' \in \mathrm{dcl}(ca_{m+1}...a_nA)$ we derive $a_1 \perp c'\,(cA)$. This completes the proof of the lemma. □

We will be interested in (types of) Grassmannians and in types of their elements, i.e. in $p_{(n)}$'s and p_c's, when p is a regular, internally isolated type. By Corollary 1.1 from [8] internal isolation is preserved under non-orthogonality of regular types whose domains differ on a finite set. Note that if $p, q \in S(A)$ are two such types which are not almost orthogonal then the names of their corresponding Grassmannians are interdefinable over A. In particular, this holds for p and $p_{(1)} = p/E_1$: the names for $\mathrm{cl}_p(\bar{a})$ and $\mathrm{cl}_{p_{(1)}}(\bar{a}/E_1)$ are interdefinable over A. Further, if such a p has successor-ordinal U-rank then, by Lemma 2.7(ii), $p_{(1)}$ has U-rank 1; for our purposes this is not an interesting case, because we are interested in isolation properties of types of U-rank ω^α when $\alpha > 1$. So, the interesting case is when p is regular and $U(p)$ is a limit ordinal. For such a type p it will turn out that p_c's are stationary for all sufficiently large n and all $c \models p_{(n)}$, and that $U(p_c)$ can be arbitrarily close to $U(p_{(1)}) = \omega^\alpha$.

Lemma 2.9. *Suppose that $p \in S(A)$ is regular, internally isolated and has limit-ordinal U-rank. Denote $p_{(1)}$ by q.*

(i) There exists a natural number n such that q_d is non-algebraic for all $m \geq n$ and all $d \models q_{(m)}$.

(ii) For any n satisfying part (i): p_c is non-algebraic for all $m \geq n$ and all $c \models p_{(m)}$.

Proof. (i) By Lemma 2.7(ii) $U(q) = \omega^\alpha > 1$ so q has a non-algebraic, forking extension $r = \mathrm{stp}(a/B)$. Let $I = (a_i \mid i \in \omega)$ be a Morley sequence in r. Then $\mathrm{Cb}(r) \subset \mathrm{dcl}(I)$. Let n be the smallest integer such that $\{a_0, ..., a_n\}$ is not independent over A. Let d be the name for $\mathrm{cl}_q(a_0...a_{n-1})$. Then $d \models q_{(n)}$ and $\mathrm{tp}(a_n/a_0, ..., a_{n-1}Ad)$ is non-algebraic, because it is a nonforking extension of r. It also extends $q_d = \mathrm{tp}(a_n/Ad)$ so q_d is non-algebraic. Now suppose that $d \leq d' \models q_{(m)}$ holds. By Lemma 2.8 we have $a_n \perp d' (dA)$ so $\mathrm{tp}(a_n/Add')$ is non-algebraic because it is a nonforking extension of q_d. On the other hand $\mathrm{tp}(a_n/Add')$ is an extension of $q_{d'}$ and $q_{d'}$ is non-algebraic, too.

(ii) p and q have interdefinable names of Grassmannians and $q \in \mathrm{dcl}(p)$ holds, so p_c is non-algebraic whenever the corresponding q_d is non-algebraic. $\square$

Definition 2.10. *Suppose that $p \in S(A)$ is regular, internally isolated and has limit-ordinal U-rank. Denote $p_{(1)}$ by q. Define n_p to be the smallest integer n such that q_d is non-algebraic for all $m \geq n$ and all $d \models q_{(m)}$.*

Remark 2.11. *Suppose that $p \in S(A)$ is regular, internally isolated and has limit-ordinal U-rank. Denote $p_{(1)}$ by q.*

(i) Strict regularity of $p_{(1)}$, established in Lemma 2.7(i), implies that $n_p \geq 2$ always holds.

(ii) By Lemma 2.9(ii) p_c is non-algebraic for any c naming a Grassmanian of p-dimension $\geq n_p$.

Lemma 2.12. *Suppose that A is finite and that $p \in S_1(A)$ is a regular, internally isolated type of limit-ordinal U-rank. Then for all $n \geq n_p$ and all $c \models p_{(n)}$:*

(i) p_c is stationary.

(ii) If $a_1, ..., a_n \in \mathrm{cl}_p(c)$ then: $\bar{a} \models p^n$ if and only if $\bar{a} \models (p_c)^n$. Moreover, the same holds for any $m \leq n$ in place of n.

(iii) $p_c^n \overset{a}{\perp} A$.

Proof. Let $q = p/E_1$ and let d be the name for $\mathrm{cl}_q(c)$. So $d \models q_{(n)}$.

(i) Suppose that a_1, a_2 realize p_c and $a_1 \downarrow a_2\,(Ac)$; we will show that $a_1 \downarrow a_2\,(A)$ holds. Otherwise $a_1 \not\downarrow a_2\,(A)$ and let $a_1/E_1 = a_2/E_1 = b \models q$. Then $a_1 \downarrow a_2\,(Ac)$ and $b \in \mathrm{dcl}(a_2 A)$ imply $a_1 \downarrow b\,(Ac)$ so, because $b \in \mathrm{dcl}(a_1 A)$, we get $b \in \mathrm{acl}(cA)$. Because c and d are interdefinable over A we get $b \in \mathrm{acl}(dA)$. This holds for any $b \models q_d$ because a_1 can be an arbitrary element of $\mathrm{cl}_p(c)$. We conclude that q_d is algebraic, which is in contradiction with $n \geq n_p$. Thus any pair of independent realizations of p_c realizes also p^2. Since $n_p \geq 2$ holds we can apply Remark 2.5(ii) to conclude that there is a unique type over cA of a pair of independent over A realizations of p_c; p_c is stationary.

(ii) Suppose that $\bar{a}$ realizes p^n and $\bar{a} \subset \mathrm{cl}_p(c)$. Let $I = b_1, b_2...$ be an infinite Morley sequence in p_c, and let $C = \mathrm{Cb}(p_c) \subset \mathrm{dcl}(I) \cap \mathrm{dcl}(cA)$. Pick the largest m such that $\{b_1, ..., b_m\}$ is independent over A; then $I \subset \mathrm{cl}_p(b_1...b_m)$ so $C \subset \mathrm{dcl}(\mathrm{cl}_p(b_1...b_m))$. Since $I \subset \mathrm{cl}_p(c)$ we have $m \leq n$. Suppose that $m < n$ holds. Then for some i we have $a_i \downarrow b_1...b_m\,(A)$. This implies $a_i \downarrow \mathrm{dcl}(\mathrm{cl}_p(b_1...b_m))\,(A)$ so $a_i \downarrow C(A)$ which is in contradiction with $a_i \models p_c$ and $C = \mathrm{Cb}(p_c)$. Therefore $m = n$ and any tuple $\bar{b} \models (p_c)^n$ also realizes p^n. This proves the direction $\Leftarrow$. The other direction follows by Remark 2.5(i).

(iii) Suppose that $p_c^n \overset{a}{\perp} A$ fails to be true. Then there is $b \downarrow c\,(A)$ such that $b \not\downarrow \bar{a}\,(Ac)$ (where $\bar{a} \models p_c^n$ and c names $\mathrm{cl}_p(\bar{a})$). Since $\mathrm{wt}_p(\bar{a}/Ac) = 0$

and $b \downarrow c\,(A)$, after replacing b by an appropriate element from $\mathrm{Cb}(\bar{a}c/bA)$, we may assume $\mathrm{wt}_p(b/A) = 0$. Then $p \perp \mathrm{stp}(b/A)$ and, because $\bar{a} \models p^n$, we have $b \downarrow \bar{a}\,(A)$. Now $c \in \mathrm{dcl}(A\bar{a})$ implies $b \downarrow \bar{a}\,(Ac)$. A contradiction. $\square$

We say that a non-algebraic type $p \in S_1(A)$ is *primitive* if there is no nontrivial A-definable equivalence relation on its locus; clearly, a primitive type is stationary. We say that a non-algebraic type $p \in S_1(A)$ is *strictly primitive* if it is stationary and for all $a, b \models p$ either $a = b$ or $a \downarrow b\,(A)$ holds; equivalently, $p^2(x, y)$ is the unique complete extensions of $p(x) \cup p(y) \cup \{x \neq y\}$ in $S_2(A)$. Clearly, a strictly regular type is strictly primitive, while a strictly primitive type is primitive.

Remark 2.13. *A primitive type is semiregular and has U-rank of the form $\omega^\alpha \cdot n$ where n is the weight of the type; in particular, it is $\not\perp$ to any of its extensions of U-rank $\geq \omega^\alpha$. Moreover, whenever p is primitive, q is regular and $p \not\perp q$, then p is q-internal and q-semiregular: Suppose that $p = \mathrm{tp}(a/A)$ is primitive and let $b \in \mathrm{dcl}(aA) \setminus \mathrm{dcl}(A)$ be such that $\mathrm{tp}(b/A)$ is semiregular with monomial U-rank; say $b = f(a)$ where f is an A-definable function. $f(x) = f(y)$ defines an equivalence relation on $p(\bar{M})$ so, because p is primitive, it is the identity relation. Thus $x \equiv a(bA) \vdash x = a$ so $a \in \mathrm{dcl}(bA)$ and $\mathrm{tp}(a/A)$ is semiregular of monomial U-rank.*

Lemma 2.14. *Suppose that $p \in S_1(A)$ is a regular, internally isolated type.*

(i) p is primitive if and only if it is strictly primitive.

(ii) If p is primitive and has limit-ordinal U-rank then p_c is strictly primitive for all $n \geq n_p$ and $c \models p_{(n)}$.

Proof. (i) If p is primitive then E_1 is the equality on $p(\bar{M})$, so any two distinct realizations of p are independent over A and p is strictly primitive.

(ii) p_c is non-algebraic by Remark 2.11(ii); it is stationary by Lemma 2.12(i). Suppose that a, b are distinct realizations of p_c. Because p is strictly primitive we have $(a, b) \models p^2$ which, by Lemma 2.12(ii), implies $(a, b) \models (p_c)^2$. Thus any pair of distinct realizations of p_c realizes $(p_c)^2$ and p_c is strictly primitive. $\square$

Remark 2.15. *If $p \in S_1(A)$ is a regular, internally isolated, primitive type of limit-ordinal U-rank then p/E_1 and p are interdefinable, so n_p is the smallest integer n for which p_c's are non-algebraic for $c \models p_{(n)}$.*

Definition 2.16. *We say that a complete type q* controls *a complete type p, or that p is q-*controlled*, if p is foreign to q (i.e. p is $\perp$ to any extension of q) and any forking extension of p is q-internal.*

Proposition 2.17. *Suppose that $p \in S_1(A)$ is a regular, primitive, internally isolated type of U-rank $\omega^{\alpha+1}$.*

(1) There exists a regular type q of U-rank ω^{α} which controls p.

(2) If $q \in S_1(A)$ has U rank ω^{α} and controls p then for all $n \geq n_p$ and $c \models p_{(n)}$ the binding group $G_c = \mathrm{Aut}_{q(\bar{M})A}(p_c(\bar{M}))$ acts transitively on the locus of $(p_c)^n$;

(3) If $c \models p_{(n)}$ and $U(p_c) \geq \omega^{\alpha}$ then the generic type of G_c is $\not\perp q$.

Proof. Without loss of generality suppose $A = \emptyset$.

(1) First we show that any forking extension of p is parallel to an extension of some p_d. Indeed, let $\mathrm{tp}(a/B')$ be a forking extension of p and let $I = a_1, a_2, ...$ be an infinite Morley sequence in $\mathrm{stp}(a/B')$. Let m be maximal such that $a_1, ..., a_m$ is independent over $\emptyset$ and let d name $\mathrm{cl}_p(a_1, ..., a_m)$. The independence of I and $d \in \mathrm{dcl}(a_1, ..., a_m)$ imply $a_{m+1} \mathop{\smile\!\!\!\!\!\!|}\, a_1...a_m d\,(B')$. Hence $\mathrm{tp}(a_{m+1}/B')$ is parallel to $\mathrm{tp}(a_{m+1}/a_1...a_m d)$ which is an extension of $\mathrm{tp}(a_{m+1}/d) = p_d$.

Let B be finite and let $q = \mathrm{tp}(a/B)$ be a stationary extension of p such that $U(q) = \omega^{\alpha}$. We will show that q controls p. p is clearly foreign to q, so it remains to prove that any forking extension of p is q-internal. Since any forking extension of p is (parallel to) an extension of p_c for some $n \geq 2$ and some $c \models p_{(n)}$, it suffices to show that any p_c is q-internal. $\mathrm{tp}(a/B)$ is a forking extension of p, so let $a_1...a_m$ and $d = a_1...a_m/E_m$ be as in the first paragraph of the proof. We have:

$$\omega^{\alpha+1} > U(p_d) \geq U(a/B) = \omega^{\alpha}.$$

p_d is clearly non-algebraic so, by Lemma 2.14(ii), p_d is primitive. By Remark 2.13 we have $U(p_d) = \omega^{\alpha} \cdot k$ where $k = \mathrm{wt}(p_d)$. Since q is parallel to an extension of p_d and $U(q) = \omega^{\alpha}$ we derive $p_d \not\perp q$. Since p_d is primitive Remark 2.13 applies and p_d is q-semiregular and q-internal.

Now let n and $c \models p_{(n)}$ be arbitrary and we will prove that p_c is q-internal. Let $b_1, ..., b_n \models p^n$ be such that $c = b_1...b_n/E_n$, and let c' be the name for $\mathrm{cl}_p(\bar{a}\bar{b})$. Then $c, d \leq c'$ and both p_d and $p_{c'}$ are primitive. In fact, $p_{c'}$ is q-semiregular. To prove it note that $p_d \,|\, dc'$ is an extension of $p_{c'}$ so $U(p_d \,|\, dc') \geq \omega^{\alpha}$ and $U(p_{c'}) = \omega^{\alpha} \cdot l$ (where $l = \mathrm{wt}(p_{c'})$) together imply $p_{c'} \not\perp p_d$. Then q-semiregularity of p_d implies $p_{c'} \not\perp q$ and $p_{c'}$ is q-internal and q-semiregular.

Now consider $p_c \,|\, c'c$. It is an extension of $p_{c'}$ so, because $p_{c'}$ is q-internal, it is q-internal, too. On the other hand, it is a nonforking extension of p_c so p_c is q-internal, too. p is q-controlled.

(2) Suppose that $q \in S(A)$ controls p and let $n \geq 2$. Since $\mathrm{tp}(c/A)$ is p-semiregular and $p \perp q$ we have $c \underset{A}{\smile} q(\bar{M})$ which, combined with $(p_c)^n \overset{a}{\perp} A$ from Lemma 2.12, implies that there is a unique type over $cAq(\bar{M})$ of a realization of $(p_c)^n$. G_c acts transitively on the locus of $(p_c)^n$.

(3) Since the action of G_c is transitive we have $U(G_c) \geq U(p_c)$ and $U(G_c) \geq \omega^\alpha$. Since G_c is q-internal and $U(q) = \omega^\alpha$, we conclude that the generic type of G_c is $\not\perp q$. $\square$

3 Proof of Theorem 1

In this section we will prove Theorem 1. For a specialist in the stable group theory the proof is rather a straightforward consequence of Proposition 2.17 and well-known facts on interpreting simple groups or fields in the superstable context. The essence is in the following: If p is NENI and $U(p) = \omega^{\alpha+1}$ then, for sufficiently large n and $c \models p_{(n)}$, Proposition 2.17 applies. We get a regular type q of U-rank ω^α which controls p, and a transitive action of G_c on the locus of $(p_c)^n$. Since p_c is strictly primitive the action is 2-transitive; in this situation it is routine to show that the α-connected component of G_c is q-connected and has trivial center. In general, for any regular type q the existence of a q-connected group with trivial center implies the existence of a q-connected simple group or of a q-connected field. In either of the cases we will conclude that q is ESN; by Lemma 3.2 this always holds in the field case, for simple groups this is an assumption of the theorem. Thus the existence of a NENI type of U-tank $\omega^{\alpha+1}$ implies the existence of an ESN type of U-rank ω^α. This suffices to produce many countable models by applying Proposition 1.1.

We will sketch the proof in some more detail assuming that the reader is familiar with the subject, references are [6] and [10]. All the groups considered are type-definable. Following [3], for a regular type p we will say that a group is p-connected if it is p-simple, connected, and has a generic type domination-equivalent to a power of p. Hrushovski's analysis of stable groups is based on the following fact (see Theorem 3.1.1 in [10]; for a group-action version see Fact 1 in [3]):

Fact 3.1. *If a generic type of a stable group G is non-orthogonal to a type p then there is a relatively definable, normal subgroup H of infinite index*

such that generic types of G/H are p-internal and $\not\perp p$.

If G is superstable and p is chosen to have minimal U-rank among types non-orthogonal to the generic of G, then (stationarizations of) generic types of G/H are domination-equivalent to a power of p and the connected component $(G/H)^0$ is p-connected. Moreover, $U(G/H) = \omega^\alpha \cdot n$ where $U(p) = \omega^\alpha$ and $n = \mathrm{wt}_p(G/H)$. As an immediate consequence we derive that the generic type of a definably simple group G is p-connected and p-internal for any regular p which is $\not\perp$ to the generic; for any such p we have $U(G) = \omega^\alpha \cdot n$ where $n = \mathrm{wt}_p(G)$.

The situation is similar with fields, one argues as in the proof of Corollary 3.1.2 from [10]: suppose that F is a superstable field whose generic is $\not\perp p$. Let H be given by Fact 3.1 applied to the additive group of F. Then $F/b\,H$ is also p-internal for every non-zero $b \in F$. $I = \bigcap_{b\neq 0} bH$ is, by Baldwin-Saxl, a finite subintersection so F/I is also p-internal. But I is an ideal of infinite index, hence trivial: F is p-internal. The conclusion is that a superstable field F is p-internal, p-semiregular and p-connected whenever p is regular and $\not\perp$ to a generic type of a field; $U(F) = \omega^\alpha \cdot n$ where $n = \mathrm{wt}_p(F)$.

Lemma 3.2. *The generic type of a superstable field of U-rank smaller than ω^ω is ESN.*

Proof. Let F be a superstable field such that $U(F) = \omega^n \cdot m$. Suppose that the generic type of F is not ESN. By Theorem 2 there exists a NENI type p which is nonorthogonal to the generic of F; without loss of generality p is over $\emptyset$. Then F is p-internal. Choose a generic $a \in F$ and a finite $B \subset F$ such that $U(a/B) = \omega^n$ and let $a'B', aB$ be a Morley sequence in $\mathrm{stp}(aB)$. Define:

$$E' = \{x \in F \mid \mathrm{wt}_p(x/BB') = 0\} \quad \text{and} \quad E = \{x \in F \mid \mathrm{wt}_p(x/B) = 0\}.$$

p is NENI so, by Lemma 2.2, both E and E' are relatively definable within F. Either of them is closed under addition and multiplication, so they are subfields of F. E is a subfield of E' and, because $a' \in E' \setminus E$, it is a proper subfield. Clearly, $U(E') < \omega^{n+1}$ and, because $a \in E$ and $U(a/B) = \omega^n$, we have $\omega^n \leq U(E), U(E') < \omega^{n+1}$. Since any superstable field is algebraically closed, E' is an infinite-dimensional vector space over E. Every element of an m-dimensional subspace is interdefinable with an element of E^m over a generic basis, so $U(E) \cdot m \leq U(E')$. Here m can be chosen arbitrarily large so $U(E') \geq \omega^{n+1}$ follows. A contradiction. $\square$

In the following, well-known fact no stability assumption is needed.

Fact 3.3. *Suppose that a group H acts faithfully and 2-transitively on an infinite set X. Then H has trivial center.*

Proof. Suppose that $Z(H)$ is nontrivial: $1 \neq h \in Z(H)$. Let $a \in X$ be such that $h(a) \neq a$ and let $b \in X$ be distinct from a and $h(a)$. 2-transitivity implies that there exists $g \in H$ mapping $(a, h(a))$ to (a, b). Then $h(g(a)) = h(a) \neq b = g(h(a))$ so g and h do not commute. A contradiction. $\square$

It is well known that the connected component of a stable group is properly defined: it is the intersection of all the relatively definable (normal) subgroups of finite index. This was generalized by Berline and Lascar in [2]: α-connected component of a superstable group G is the intersection of all relatively definable (normal) subgroups H such that $U(G/H) < \omega^\alpha$; denote it by G^α. Then G^α is the smallest type-definable subgroup whose index has U-rank $< \omega^\alpha$. However, the meaning of 'q-connected component of a group' is not clear at all in the general stable case; it requires some additional assumptions.

Below we will be interested in groups which are q-internal where q is regular and has U-rank ω^α. For such a group G we have $U(G) = \omega^\alpha \cdot m + \xi$ where $\xi < \omega^\alpha$. Here $U(G^\alpha) = \omega^\alpha \cdot m$ and $m = \mathrm{wt}_q(G) = \mathrm{wt}_q(G^\alpha)$. G^α is q-connected and it is the largest q-connected subgroup of G. Therefore, q-connected subgroups of G are precisely those which are α-connected.

Proposition 3.4. *Suppose that $q \in S_1(A)$ is a regular type of U-rank ω^α which controls a primitive, NENI type $p \in S_1(A)$ of U-rank $\omega^{\alpha+1}$. Then there exists a simple, q-connected group or a q-connected field.*

Proof. Without loss of generality assume $A = \emptyset$. Fix $n \geq n_p$ sufficiently large and $c \models p_{(n)}$ so that $U(p_c) \geq \omega^\alpha$ and Proposition 2.17(3) applies: the binding group G_c is $\not\perp q$. Since G_c is q-internal $U(G_c) = \omega^\alpha \cdot m + \xi$ where $\xi < \omega^\alpha$. Let $H \leq G_c$ be the α-connected component of G_c.

We *claim* that H acts transitively on the locus of $(p_c)^n$. By Proposition 2.17 G_c acts transitively on the locus of $(p_c)^n$, so G_c/H acts transitively on the set of H-orbits. $U(G_c/H) < \omega^\alpha$ implies that the U-rank of any (name of an) orbit is $< \omega^\alpha$. Let d be a name of such an orbit. Clearly, d is in the dcl of some $\bar{a} \models (p_c)^n$. By Remark 2.13(i) p_c is semiregular. It is also q-internal because p is q-controlled, so $U(p_c) \geq \omega^\alpha$ implies $U(\bar{a}) = \omega^\alpha \cdot k$. Then $U(d) < \omega^\alpha$ and $d \in \mathrm{dcl}(\bar{a})$ imply $U(d) = 0$, so there are only finitely many orbits. Because $(p_c)^n$ is stationary, there is a unique H-orbit and H acts transitively on $(p_c)^n$, proving the claim.

p is a primitive, NENI type so Lemma 2.14(ii) applies: p_c is strictly primitive. Transitivity of the action of H on $(p_c)^2(\bar{M})$ and strict primitivity

of p_c imply that H acts 2-transitively on $p_c(\bar{M})$. By Fact 3.3 H has trivial center. Altogether: H is q-internal, q-connected and has trivial center.

Now, suppose that G is a q-internal, q-connected group of minimal q-weight having trivial center. Clearly, G is non-abelian. There exists a series of normal, relatively definable subgroups of $G = G_0 > G_1 > ... > G_n = \{1\}$ such that each quotient G_i/G_{i+1} is either abelian or simple (this is a consequence of the Zilber Indecomposability Theorem, see Corollary 3.6.15 in [10]). Then, because G is non-abelian and q-connected, we have $\mathrm{wt}_q(G_1) < \mathrm{wt}_q(G)$. Since G is q-connected G/G_1 is q-connected, too. Now we have two cases: G/G_1 is either simple or abelian. In the first we are done, so suppose that G/G_1 is abelian and we will find a q-connected field.

Since G/G_1 is abelian the commutator subgroup G' is a proper subgroup of G; also, it is relatively definable in G and q-connected (again by indecomposability, see Corollary 3.6.13 in [10]). The minimality of $\mathrm{wt}_q(G)$ implies that G' has non-trivial center: $Z(G')$ is non-trivial and G-invariant. Let K be a G-minimal (minimal, type-definable, nontrivial, G-invariant) subgroup of $Z(G')$. First we rule out the possibility $U(K) < \omega^\alpha$: if it holds then the U-rank of any G-orbit in K is $< \omega^\alpha$, so $[G : C_G(k)] < \omega^\alpha$ holds for all $k \in K \setminus \{1\}$. Since G is α-connected we have $G = C_G(k)$ and k is central in G. A contradiction. Therefore $U(K) \geq \omega^\alpha$ and K^α, the α-connected component of K, is non-trivial. For any $g \in G$ we have $U(K/gK^\alpha) = U(K/K^\alpha) < \omega^\alpha$, which implies ($gK^\alpha \supseteq K^\alpha$ and, similarly, $g^{-1}K^\alpha \supseteq K^\alpha$ so) $gK^\alpha = K^\alpha$. Thus K^α is G-invariant and α-connected. Because K is G-minimal we have $K = K^\alpha$ and K is α-connected.

Let $C_G(K)$ be the pointwise centralizer of K. It is a relatively definable, normal subgroup of G and it contains G' (because $K \subset Z(G')$): $G/C_G(K)$ is abelian. Also, because G is centerless, $C_G(K)$ is not the whole of G. We conclude that $G/C_G(K)$ is non-trivial and, because G is q-connected, $G/C_G(K)$ is q-connected, too. Further, for any $g \notin C_G(K)$ there exists $k \in K$ such that $g(k) \neq k$; it follows that $G/C_G(K)$ acts faithfully on K. Since the orbits under the action of G and $G/C_G(K)$ on K are the same, K is a $G/C_G(K)$-minimal, q-connected abelian group. Therefore we have a faithful action of a q-connected, abelian group $G/C_G(K)$ on the abelian, q-connected, $G/C_G(K)$-minimal group K. In this situation Zilber's Indecomposability Theorem implies that K is the additive group of a field: see Theorem 5.3.1 in [10], or (the proof of) Lemma 2 from [3]. K is a q-connected field. This completes the proof of the proposition. $\square$

Proof of Theorem 1. Suppose that $U(T) \geq \omega^\omega$ holds and that the generic type of any definable, infinite, simple group is ESN. We will prove that T

has $2^{\aleph_0}$ countable models. By Proposition 1.1 it suffices to find an infinite $I \subset \omega$ and a family $\{p_n | n \in I\}$ of regular, ESN types over finite domains such that $U(p_n) = \omega^n$ holds for all $n \in I$. Suppose, for a contradiction, that such a family does not exist; let n be such that any regular type of U-rank ω^m for $m \geq n$ is not ESN. Then, by Theorem 2, any such type is non-orthogonal to a NENI type. Fix a NENI type p' of U-rank ω^{n+1} and, without loss of generality, assume that $p' \in S_1(\emptyset)$ is primitive (by Remark 2.13, say).

Now we apply Proposition 2.17: there exists a finite set A and a regular type $q \in S_1(A)$ which controls p' and has U-rank ω^n. Since $p' \mid A$ is NENI, by Remark 2.13, $p = (p'|A)/E_1$ is a primitive, NENI type of U-rank ω^{n+1}. Any forking extension of p is q-internal because it is in the dcl of some forking extension of p', and the latter extension is q-internal because p' is q-controlled. Hence q controls p. We have the following situation: $p, q \in S_1(\emptyset)$ are regular, p is a primitive NENI type, $U(p) = \omega^{n+1}$, $U(q) = \omega^n$ and q controls p. By Proposition 3.4 there exists a q-connected, simple group or a q-connected field. By our assumption on generic types of simple groups and Lemma 3.2, in either case the generic type is ESN; by Theorem 2 q is ESN. A contradiction. $\square$

It was conjectured in [9] that the answer to Question 1 is affirmative. Here we will be a little bit more careful:

Conjecture 2. *The generic type of a simple superstable group of U-rank $\omega^{\alpha+1} \cdot n$ is ESN.*

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